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Hom_{*} commuting with filtered products¹

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Abstract: In a sufficiently rich category, such as a category of R -modules, and a given infinite cardinal κ , we examine classes \mathcal{H}_*^κ of objects M , such that a natural monomorphism is an isomorphism:

$$\prod_{i \in I}^\kappa \operatorname{Hom}(M, A_i) \cong \operatorname{Hom}(M, \prod_{i \in I}^\kappa A_i),$$

for every family of objects $\{A_i : i \in I\}$ (\prod^κ denotes the subproduct of all vectors with support $< \kappa$).

§1. Preliminaries.

We will assume the axiom of choice or equivalently that every set may be well ordered; one consequence is the existence of arbitrary infinite products in the category of sets. Furthermore we will assume that the categories we work with have arbitrary products and coproducts. We will identify any cardinal κ with the smallest (initial) ordinal of that cardinality, when it is convenient to do so; this is equivalent to the statement that every cardinal is of the form \aleph_α , for some ordinal number α ; we will use ‘fin’ to denote a finite cardinal. Thus, an arbitrary non-empty index set I may be assumed to be well ordered by an (initial) ordinal; if needed, we will assume ordinals to be regular, i.e. that $\operatorname{cf} I = I$. A cardinal κ is regular, if it is not singular, i.e., if it cannot be represented as $\sup\{\alpha_i : i < \theta\}$, where each α_i, κ and $\theta < \kappa$ is a limit ordinal. Equivalently, κ is regular iff it cannot be represented as the sum of less than κ smaller cardinals. $|I|^+$ will denote the successor cardinal to cardinal $|I|$. An infinite cardinal \aleph_α is a successor cardinal, if it is of the form $\aleph_{\beta+1} = \aleph_\alpha$; if α is a limit ordinal, then \aleph_α is called a limit cardinal. Every successor ordinal is regular, but it is not always the case with limit ordinals, which may be singular. In this study, we are mostly interested in regular ordinals (cardinals).

If $a = (a_i)_{i \in I} \in \prod_{i \in I} A_i$, we will also write a as a formal sum $\sum_{i \in I} a_i$ which is, more precisely, the formal sum $\sum_{i \in I} p_i a_i$, where $p_i : A_i \rightarrow \prod A_i$ are the natural product injections. In absence of a topology/metric, the sum will “make sense” when there are only totality of finitely many non-zero coordinates, at every coordinate of all I -vectors being summed. This will always be the case if we are doing genuine summation, not just the formal one.

¹The first version of this paper, dated February 11, 2007, was communicated to C. U. Jensen

In essence, our note concerns the category of unital (one-sided) R -modules, but we are using the language of general categories to indicate that the results and the proofs carry over to this more general setting, *mutatis mutandis*.

For an arbitrary family $\{A_i, i \in I\}$ of (non-zero) objects, and an arbitrary infinite cardinal κ , $\prod_{i \in I}^\kappa A_i$ will denote the filtered κ -product, namely it consists of all the vectors with support $< \kappa$. The natural κ -product-to-product (κ ptp) embedding will be denoted by u_κ or $u : \prod_{i \in I}^\kappa A_i \longrightarrow \prod_{i \in I} A_i$.

For $\kappa \geq |I|^+$ ($\kappa = \aleph_0$) we have respectively

$$\prod_{i \in I}^\kappa A_i = \prod_{i \in I} A_i \quad \left(\prod_{i \in I}^\kappa A_i = \prod_{i \in I} A_i \right).$$

Fact: For every object M , we have a natural monomorphism

$$\phi : \prod_{i \in I}^\kappa \text{Hom}(M, A_i) \longrightarrow \text{Hom}(M, \prod_{i \in I}^\kappa A_i) \quad (*)$$

given by $(f_i : M \longrightarrow A_i)_{i \in I} \mapsto f = \sum_{i \in I} p_i f_i : M \longrightarrow \prod_{i \in I}^\kappa A_i$ (with coordinates $\forall \pi_i f = f_i$). This monomorphism is an isomorphism in case $\kappa \geq |I|^+$, as well as when I is a finite index set, regardless of κ , thus, we can assume in the sequel, when needed, without loss of generality, that $\aleph_0 \leq \kappa \leq |I|$. When $|I|$ is an infinite cardinal and $\kappa \leq |I|$, then we do not necessarily have an isomorphism. For instance, if $\kappa = \aleph_0$, $(*)$ is not an isomorphism, for every M and every family of non-zero modules A_i , even when all $A_i = R$. Thus we have the following

Task 1. Investigate, for various infinite cardinals κ , and if possible characterize, objects M such that for every infinite index set I , every family of (non-zero) objects $\{A_i, i \in I\}$ monomorphism $(*)$ is an isomorphism. Call every such object M a *Hom $_*$ - κ -commuting object*. Given an infinite cardinal κ , denote by \mathcal{H}_*^κ the class of all Hom $_*$ - κ -commuting objects. For $\kappa = \aleph_0$ the Hom $_*$ - κ -commuting object was introduced by Mitchell, no later than 1965, under the name *small object* (see e.g. also Rentchler, 1969 where small objects go under the name of Σ -type object (or Σ -generated) object).

§2. Equivalent definitions.

While this is an ambitious task, we show in the sequel how to arrive at a number of illuminating results. We begin with the following:

Lemma 1. *Given an infinite cardinal κ , an additive category \mathcal{C} with arbitrary coproducts and products, an arbitrary non-empty index set I and a morphism $f : M \longrightarrow \prod_{i \in I}^\kappa A_i$, the following statements are equivalent:*

- (1) *The morphism f factors through a κ' -subproduct, $\kappa' < \kappa$, namely there is a subset $J \subseteq I$, $|J| = \kappa' < \kappa$ and a factorization $M \xrightarrow{f'} \prod_{i \in J}^\kappa A_i \xrightarrow{p_J} \prod_{i \in I}^\kappa A_i$,*

- $f = p_J f'$ (here p_J is the natural embedding of smaller κ -product into the larger; note that $\prod_{i \in J}^\kappa A_i = \prod_{i \in J} A_i$, since $|J| < \kappa$).
- (2) $f = \sum_{i \in J} p_i f_i$, for some $J \subseteq I$, $|J| < \kappa$ and for some morphisms $f_i : M \rightarrow A_i$, $i \in J$.
 - (3) If, for some $J \subseteq I$, $|J| < \kappa$, $q_{I \setminus J} : \prod_{i \in I}^\kappa A_i \rightarrow \prod_{i \in I \setminus J}^\kappa A_i$ is the canonical quotient map $= \pi_{I \setminus J} | \prod_{i \in I}^\kappa A_i$, then $q_{I \setminus J} f = 0$.
 - (4) There is a subset $J \subseteq I$, $|J| < \kappa$ such that $\pi_{I \setminus J} u f = 0$.

Proof As before, denote by π'_i and p'_i the natural product projections and injections associated with the product indexed by J , and likewise by u' the corresponding κ tp-morphism. Note that

$$(*) \forall i \in J p_J p'_i = p_i \text{ and } (**) \forall i \in J \pi'_i u' = \pi_i u p_J.$$

The proof is as follows:

$$(1) \Rightarrow (2): f = p_J f' = p_J (\sum_{i \in J} p'_i \pi'_i u') f' = (\text{by } (*)) = \sum_{i \in J} p_i \pi'_i u' f' = (\text{by } (**)) \\ = \sum_{i \in J} p_i \pi_i u p_J f' = \sum_{i \in J} p_i \pi_i u f; \text{ denote } f_i = \pi_i u f, \text{ for all } i \in J.$$

$$(2) \Rightarrow (3): q_{I \setminus J} f = q_{I \setminus J} (\sum_{i \in J} p_i f_i) = \sum_{i \in J} q_{I \setminus J} p_i f_i = 0.$$

(3) \Rightarrow (4): If u'' denotes the κ tp map associated with the κ product on the index set $I \setminus J$, then the proof is established by noting that $u'' q_{I \setminus J} = \pi_{I \setminus J} u$ and thus $\pi_{I \setminus J} u f = u'' q_{I \setminus J} f = 0$. In fact the same observation proves the reverse implication.

(4) \Rightarrow (1): Equality $\pi_{I \setminus J} u f = 0$ ensures $\text{Im } f \subseteq \text{Ker } (\pi_{I \setminus J} u) \subseteq \prod_{i \in J}^\kappa A_i$ and this in turn ensures validity of (1) \square

We extend this result as follows, by not assuming a priori that index sets are the same or that the components A_i are the same, in each of the equivalent statements:

Proposition 2. *Given an additive category \mathcal{C} with arbitrary (κ -products and) co-products and an object M , the following are equivalent:*

- (1) For every non-empty index set I and an arbitrary family of objects $\{A_i : i \in I\}$ in \mathcal{C} , every morphism $f : M \rightarrow \prod_{i \in I}^\kappa A_i$ in \mathcal{C} factors through a κ' -subproduct, $\kappa' < \kappa$, namely there is a subset $J \subseteq I$, $|J| = \kappa' < \kappa$ and a factorization $M \xrightarrow{f'} \prod_{i \in J}^\kappa A_i \xrightarrow{p_J} \prod_{i \in I}^\kappa A_i$, with $f = p_J f'$.
- (2) For every non-empty index set I and an arbitrary family of objects $\{A_i : i \in I\}$ in \mathcal{C} and every morphism $f : M \rightarrow \prod_{i \in I}^\kappa A_i$ in \mathcal{C} , there is a $J \subseteq I$, $|J| < \kappa$, such that $f = \sum_{i \in J} p_i f_i$, for some morphisms $f_i : M \rightarrow A_i$, $i \in J$.
- (3) For every non-empty index set I and an arbitrary family of objects $\{A_i : i \in I\}$ in \mathcal{C} and every morphism $f : M \rightarrow \prod_{i \in I}^\kappa A_i$ in \mathcal{C} there is a $J \subseteq I$, $|J| < \kappa$, such that $q_{I \setminus J} f = 0$, i.e. $\text{Im } f \subseteq \prod_{i \in J}^\kappa A_i$.
- (4) The functor $\text{Hom}_{\mathcal{C}}(M, -)$ commutes with κ -products, i.e., for every non-empty index set I and an arbitrary family of objects $\{A_i : i \in I\}$

$$\phi : \prod_{i \in I}^\kappa \text{Hom}(M, A_i) \rightarrow \text{Hom}(M, \prod_{i \in I}^\kappa A_i) \quad (*)$$

via the natural isomorphism of Abelian groups $\phi : (f_i)_{i \in I} \mapsto \sum_{i \in I} p_i f_i$.

Proof Lemma 1 establishes equivalence of the first three statements, since, a posteriori, it turns out that the index set I and the product components A_i may be the same in each of the equivalent statements.

(2) \Rightarrow (4): Given a morphism $f : M \longrightarrow \prod_{i \in I}^\kappa A_i$ in \mathcal{C} , there is a $J \subseteq I$, $|J| < \kappa$ such that $f = \sum_{i \in J} p_i f_i$, for some morphisms $f_i : M \longrightarrow A_i$. Note now that $h = (f_{i0})_{i \in I} \in \prod_{i \in I}^\kappa \text{Hom}_{\mathcal{C}}(M, A_i)$ with $f_{i0} = f_i$, for $i \in J$ and $f_{i0} = 0$ for $i \in I \setminus J$ is such that $\phi(h) = f$.

(4) \Rightarrow (2): Given a morphism $f : M \longrightarrow \prod_{i \in I}^\kappa A_i$ in \mathcal{C} , there is an $h = (f_i)_{i \in I} \in \prod_{i \in I}^\kappa \text{Hom}_{\mathcal{C}}(M, A_i)$ where, for some $J \subseteq I$, $|J| < \kappa$, $f_i = 0$ whenever $i \in I \setminus J$ and $\phi(h) = f$, i.e. $\sum_{i \in J} p_i f_i = f$. \square

Finally, we have the following series of equivalent properties that could be used to define $\text{Hom}_*\text{-}\kappa$ -commuting objects:

Theorem 3. *In an additive category with infinite products and coproducts, given an infinite (regular) cardinal κ , the following are equivalent, for an object M :*

- (1) *for every infinite set I , and every family of objects $\{A_i : i \in I\}$, the natural monomorphism defined in (*) is an isomorphism:*

$$\prod_{i \in I}^\kappa \text{Hom}(M, A_i) \cong \text{Hom}(M, \prod_{i \in I}^\kappa A_i) \quad (*I)$$

(arbitrary Hom definition);

- (2) *M is a $\text{Hom}_*\text{-}\kappa$ -commuting with families of cardinality κ , i.e., for every index set J with $|J| = \kappa$, and, for an arbitrary family of objects $\{A_i : i \in J\}$,*

$$\prod_{i \in J}^\kappa \text{Hom}(M, A_i) \cong \text{Hom}(M, \prod_{i \in J}^\kappa A_i) \quad (*\kappa)$$

via the natural isomorphism of Abelian groups

(κ -Hom definition);

- (3) *for an arbitrary non-empty index set I , for every family of objects $\{A_i : i \in I\}$, and every morphism $f : M \longrightarrow \prod_{i \in I}^\kappa A_i$, there is a $J' \subseteq I$ with $|J'| < \kappa$, such that for all $i \in I \setminus J'$, $\pi_i u f = 0$*

(arbitrary coordinatewise definition);

- (4) *for every family $\{A_i, i \in J\}$, $|J| = \kappa$, of objects, and every morphism $f : M \longrightarrow \prod_{i \in J}^\kappa A_i$, $\pi_i u_J f = 0$, for all $i \in J \setminus J'$ for some $J' \subseteq J$, $|J'| < \kappa$.*

(κ -coordinatewise definition);

- (5) *for every infinite index set I , for every family $\{A_i : i \in I\}$ of objects, and every morphism $f : M \longrightarrow \prod_{i \in I}^\kappa A_i$, there is a well-ordering on I such that, there is an $i_0 \in I$, $i_0 < \kappa$ with $\pi_{i > i_0} f = 0$ (arbitrary tailwise definition);*

- (6) *for every family $\{A_i\}_{i \in J}$ of objects, with $|J| = \kappa$ (J well-ordered), and every morphism $f : M \longrightarrow \prod_{i \in J}^\kappa A_i$, there is an $i_0 \in J$ with $i_0 < \kappa$ and $\pi_{i > i_0} f = 0$ (κ -tailwise definition).*

(5) and (6) need regularity of κ .

Proof Note that the index sets in (1), (3) and (5) are arbitrary, unlike (2), (4), (6) where index sets are of cardinality κ . Thus, each of (1),(3),(5) implies respectively (2),(4),(6). Equivalence of (1), (3) and (5) (arbitrary index sets) and equivalence of (2), (4) and (6) follow from Lemma 1 and Proposition 2. We only need to prove that one of the even numbered statements implies any of the odd numbered ones, to complete the proof of equivalence of all the statements. First, we show that regularity of κ is needed when working with tailwise definitions.

(3) \Rightarrow (5): Assume that $f : M \longrightarrow \prod_{i \in I}^{\kappa} A_i$ is an arbitrary morphism. By (3), there is a $J' \subseteq I$ with $|J'| < \kappa$, such that for all $i \in I \setminus J'$, $\pi_i u f = 0$. We can well order I in such a way that $|J'|$ is its initial segment (of cardinality $< \kappa$). Because, κ is regular, there is an $i_0 < \kappa$ in I with $\pi_i u f = 0$, for all $i > i_0$, which is the same as $\pi_{i > i_0} f = 0$. This completes the proof of all the equivalences.

(4) \Rightarrow (6): The proof is the same, mutatis mutandis, as for (3) \Rightarrow (5).

(2) \Rightarrow (4): Let $f : M \longrightarrow \prod_{i \in J}^{\kappa} A_i$ be an arbitrary morphism in \mathcal{C} . By the assumption, isomorphism $(*\kappa)$ holds, hence we can find morphisms $f_i : M \longrightarrow A_i$, with $\text{supp}(f_i)_{i \in J} < \kappa$ i.e. there is a $J' \subseteq J$ with $|J'| < \kappa$ with $f_i = 0$, for all $i \in J \setminus J'$ and such that $f = \sum_{i \in J} p_i f_i$ (summation of $< \kappa$ non-zero summands, indexed by J'). Then $\forall i \in J \setminus J'$, $\pi_i u_J f = \pi_i \sum_{i \in J} u_J p_i f_i = \pi_i (\sum_{i \in J \setminus J'} u p_i f_i + \sum_{i \in J'} u p_i f_i) = 0$.

(4) \Rightarrow (3): It is only non-trivial to consider cases when $|I| > \kappa$. If, on the contrary, there is a $J \subseteq I$ with $|J| = \kappa$, such that $\forall i \in J$, $\pi_i u f \neq 0$, then we consider the cut of f : $g = \pi_J f : M \longrightarrow \prod_{i \in J}^{\kappa} A_i$. By (4), there exists a $J' \subseteq J$ with $|J'| < \kappa$ and $\forall i \in J \setminus J'$, $\pi_i u g \neq 0$, which would be a contradiction. \square

§3. Examples, a characterization and constructions.

$\text{gen } M$ will denote the cardinality of a minimal set of generators of M (and sometimes such a set of generators itself).² and the contravariant Hom functors.

Proposition 4. *Let $\kappa \leq |I|$ be an infinite cardinal and let M be an R -module with $\text{gen } M < \kappa$. Then the natural monomorphism*

$$\phi : \prod_{i \in I}^{\kappa} \text{Hom}(M, A_i) \longrightarrow \text{Hom}(M, \prod_{i \in I}^{\kappa} A_i) \quad (*)$$

is an isomorphism, for every family of objects $A_i, i \in I$, namely M is a $\text{Hom}_ - \kappa$ -commuting object. Thus, if $\text{gen } M < \kappa$, then $M \in \mathcal{H}_*^{\kappa'}$, for every $\kappa' \geq \kappa$. In particular, every finitely generated R -module M is $\text{Hom}_* - \kappa$ -commuting, for every infinite κ .*

Proof The reason for surjectivity is that $\forall f : M \longrightarrow \prod_{i \in I}^{\kappa} A_i$, the image $\text{gen Im } f \leq \text{gen } M < \kappa$, and we may assume $\text{gen } f(M) = \{a^j = (a_i^j)_{i \in I}, j \in J\}$ with $|J| < \kappa$ and

²This practical notation was first introduced in Dimitric, 1984 and it has since been adopted in a number of cases, by other authors, without reference

support of every $a^j < \kappa$. For every $i \in I$, we define $f_i : M \rightarrow A_i$ as follows: For $m \in M$, let $f(m) = \sum_{j \in J_m} r_j a^j =$ (each $|J_m| \leq |J|$ is finite) $\sum_{j \in J_m} r_j (a_i^j)_{i \in I} = \left(\sum_{j \in J_m} r_j a_i^j \right)_{i \in I}$ (finite sums); define $\forall i \in I$ $f_i(m) = \sum_{j \in J_m} r_j a_i^j$. $\text{Supp } (f_i)_{i \in I} < \kappa$ since $\forall m \in M$ $\text{supp } f(m) < \kappa$ and $|J| < \kappa$. Clearly, by definition, $\phi(f_i)_{i \in I} = f$, which proves surjectivity. \square

When $\kappa = \aleph_0$, we get the fact that, if M is a finitely generated object, then $\text{Hom}(M, -)$ commutes with countable coproducts.

Task 2. Find and characterize categories (rings R) such that the only $\text{Hom}_* - \kappa$ -commuting modules are those that are $< \kappa$ -generated, as well as those where there are $\text{Hom}_* - \kappa$ -commuting modules generated by at least κ elements.

Proposition 5. *Let κ be any infinite limit cardinal and $M \in R\mathbf{Mod}$ be such that no ascending (smooth) κ -chain of proper submodules of M like $M_0 < M_1 < \dots < M_\alpha < \dots < M$, $\alpha < \rho \leq \kappa$, fills the whole of M , i.e. $\cup_{\alpha < \rho} M_\alpha = \sum M_\alpha \neq M$. Then M is $\text{Hom}_* - \kappa$ -commuting.*

Proof By Theorem 3, it is sufficient to prove this for index sets I with $\kappa = |I|$; we well order I so that I represents the smallest ordinal of cardinality κ . For every $\alpha < \kappa$ we will denote by $\Pi_\alpha = \prod_{i \in I}^{\alpha\kappa} A_i$ the truncated κ -product that consists of elements $(a_i)_{i \in I} \in \Pi^\kappa = \prod_{i \in I}^\kappa A_i$ with $a_i = 0$, for all $i > \alpha$. Note that $\{\Pi_\alpha\}_{\alpha < \kappa}$ is a smooth κ -chain with union Π^κ . Let $f \in \text{Hom}(M, \prod_{i < \kappa}^\kappa A_i)$; denote by $M_\alpha = f^{-1}(\Pi_\alpha)$ (the inverse image of the α -truncated k -product of A_i 's). Since $\{\Pi_\alpha\}_{\alpha < \kappa}$ is a smooth κ -chain uniting in Π^κ , $\{M_\alpha\}_{\alpha < \kappa}$ is likewise a smooth κ -chain uniting in $M = f^{-1}(\Pi^\kappa)$. By the assumption, this may happen only if not all the links are proper subobjects of M , i.e., if there exists an $\alpha < \kappa$ with $M_\alpha = f^{-1}(\Pi_\alpha) = M$. This means that $f : M \rightarrow \Pi_\alpha = \prod_{i \in I}^{\alpha\kappa} A_i$. But $\prod_{i \in I}^{\alpha\kappa} A_i \cong \prod_{i \leq \alpha} A_i$ and, in this case $f = \sum_{i < \alpha} p_i f_i$, for some $f_i : M \rightarrow A_i$, which proves the statement. Note that the same proof goes if it is done by transfinite induction on κ . \square

Proposition 6. *For an arbitrary infinite cardinal κ , let M be $\text{Hom}_* - \kappa$ -commuting. Then: No strictly ascending (smooth) ρ -chain $\rho \leq \kappa$ of proper submodules of M like $M_0 < M_1 < \dots < M_\alpha < \dots < M$, $\alpha < \rho \leq \kappa$ fills the whole of M , i.e. $\cup_{\alpha < \rho} M_\alpha = \sum M_\alpha \neq M$.*

Proof (here $\rho = \kappa$) Assume that M is $\text{Hom}_* - \kappa$ -commuting and suppose, that on the contrary, for some such chain, we have $\cup_{i < \kappa} M_i = \sum M_i = M$. Define $f(x) = (x + M_i)_{i < \kappa}$ and hope that f is a morphism $f : M \rightarrow \prod_{i \in \kappa} M/M_i$. For every $x \in M$, there is the smallest $i_x < \kappa$ with $x \in M_{i_x}$ and then, $\forall i' \geq i_x$, $x \in M_{i'}$ which implies that $|\text{supp } f(x)| \leq |i_x| < \kappa$, thus, indeed $f \in \text{Hom}(M, \prod_{i \in \kappa} M/M_i)$. Then, by the assumption, $f = (f_i)_{i < \kappa}$, for some $(f_i)_{i < \kappa} \in \prod_{i < \kappa}^\kappa \text{Hom}(M, M/M_i)$. This means that there exists a $\kappa_1 < \kappa$, such that $\forall i > \kappa_1$ $f_i = 0$. This means that $\forall i > \kappa_1$ $\forall x \in M$, $x \in M_i$. This is a contradiction, since it is assumed that all M_i are proper submodules of M . Hence, our claim holds. \square

Corollary 7. *Let κ be an infinite cardinal and $M \in R\mathbf{Mod}$ such that $\text{gen } M = \kappa$. Then,*

- (1) *If κ is a limit cardinal, M is not Hom_* - κ -commuting .*
- (2) *Furthermore, if κ is a limit cardinal, $\text{gen } M \leq \kappa$ and M is Hom_* - κ -commuting, then $\text{gen } M < \kappa$.*

Proof (1) The reason is as follows: As usual, κ denotes the initial ordinal representing cardinal κ . If $\text{gen } M = \{m_\alpha : \alpha < \kappa\}$, denote $M_\alpha = \langle \{m_i : i \leq \alpha\} \rangle$. Then $M_\alpha, \alpha < \kappa$ is an ascending κ -chain of proper subobjects of M , with $\cup_{\alpha < \kappa} M_\alpha = M$, because κ is a limit cardinal. By Proposition 6, this would be impossible, if M were to be Hom_* - κ -commuting .

(2) This is a reformulation of (1). □

In any Abelian category with products and coproducts, we have the following:

Proposition 8. *Let κ be an infinite (regular) cardinal. Then: zero object?*

- (1) *The 0 object is Hom_* - κ -commuting , thus the class of Hom_* - κ -commuting objects in any category with the zero object is non-empty. Moreover all the $< \kappa$ -generated objects are in \mathcal{H}_*^κ .*
- (2) *An object is in \mathcal{H}_*^κ iff all its quotient objects are in \mathcal{H}_*^κ .*
- (3) *If $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is an exact sequence and A, C are in \mathcal{H}_*^κ objects, then B is likewise in \mathcal{H}_*^κ .*
- (4) *For an (infinite) index set I , and an arbitrary family of objects $\{B_i : i \in I\}$, the κ -product $M = \prod_{i \in I}^\kappa B_i$ is in \mathcal{H}_*^κ , iff only finitely many $B_i \neq 0$.*
- (5) *Given a finite ascending sequence $0 = A_0 < A_1 < \dots < A_n = C$ of subobjects of an object C , such that all the factors A_{i+1}/A_i , $i = 0, 1, \dots, n-1$ are in \mathcal{H}_*^κ , then C is likewise in \mathcal{H}_*^κ . The claim is no longer true, if the length of chain is infinite (such as of countable cofinality).*
- (6) *Let $F : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ be an equivalence of two categories with κ -products and $M \in \mathcal{A}_1$. Then M is Hom_* - κ -commuting in \mathcal{A}_1 if and only if $F(M)$ is Hom_* - κ -commuting in \mathcal{A}_2 .*

Proof (1) is trivial, in view of Proposition 4.

(2) holds, because a morphism $f : M/D \longrightarrow \prod_{i \in I}^\kappa A_i$ gives rise to a morphism $f q : M \longrightarrow \prod_{i \in I}^\kappa A_i$ (q is the canonical quotient map). If M is Hom_* - κ -commuting then, by Proposition 2, $f q$ is expressible as a sum $\sum_{i \in J} p_i f_i$, for some $|J| < \kappa$ and some morphisms $f_i : M \longrightarrow A_i$. Given $\bar{x} \in M/D$, define, $\forall i \in I$, $\bar{f}_i : M/D \longrightarrow A_i$ by $\bar{f}_i(\bar{x}) = f_i(x)$. \bar{f}_i are morphisms since f_i and q are; moreover every \bar{f}_i is well-defined, for if $\bar{x} = \bar{x}'$, then $f(\bar{x}) = f q(x) = f q(x') = f(\bar{x}')$ hence $\sum p_i f_i(x) = \sum p_i f_i(x')$ and thus $\sum p_i f_i(x - x') = 0$, $i \in J$. The latter is in $\prod_{i \in I}^\kappa A_i$, since $|J| < \kappa$ thus every $p_i f_i(x - x') = 0$ and this is possible only if for every $i \in I$, $f_i(x) = f_i(x')$. We can now represent $f = \sum p_i \bar{f}_i$ as a J -sum which, by Proposition 2, means that the quotient M/D is Hom_* - κ -commuting . The other implication is trivial (once we take $D = 0$).

(3) (Draw the commutative diagram to follow the argument easier). Consider an arbitrary morphism $f : B \rightarrow \prod_{i \in I}^\kappa A_i \in \mathcal{C}$. Then $f\alpha : A \rightarrow \prod_{i \in I}^\kappa A_i$ and A is assumed to be in \mathcal{H}_*^κ , hence there is a $J \subseteq I$, $|J| < |I|$ such that $q_{I \setminus J} f \alpha = 0$, by Proposition 2.(3). Denote $\chi = q_{I \setminus J} f$. By the universal property of the quotient (cokernel) construction, there is a unique $\gamma : C \rightarrow \prod_{i \in I \setminus J}^\kappa A_i$ such that $\gamma\beta = \chi$. However C is assumed to be in \mathcal{H}_*^κ , hence, there is a $J' \subseteq I \setminus J$, $|J'| < \kappa$ with $q'_{I \setminus J \setminus J'} \gamma = 0$, where $q'_{I \setminus J \setminus J'} : \prod_{i \in I \setminus J}^\kappa A_i \rightarrow \prod_{i \in I \setminus J \setminus J'}^\kappa A_i$ is the map corresponding to γ , via Proposition 2.(3). This implies $0 = q'_{I \setminus J \setminus J'} \gamma \beta = q'_{I \setminus J \setminus J'} q_{I \setminus J} f = q_{I \setminus (J \cup J')} f$. Since both $|J|, |J'| < \kappa$, so is their union and $q_{I \setminus (J \cup J')} f = 0$, which, by Proposition 2.(3) establishes the fact that B is in \mathcal{H}_*^κ .

(4) If only finitely many components are $\neq 0$, M is clearly $\text{Hom}_*\text{-}\kappa$ -commuting, by way of canonical isomorphisms for finite products/coproducts. Assume for a moment that there are infinitely many $B_i \neq 0$ (say countably many). Then $M_{\mathbb{N}} = \prod_{i \in \mathbb{N}}^\kappa B_i$ is the ascending union of its proper subobjects $M_n = \prod_{i \in n}^\kappa B_i$, $n \in \mathbb{N}$. By Proposition 6 this means that $\prod_{i \in \mathbb{N}}^\kappa B_i$ is not $\text{Hom}_*\text{-}\kappa$ -commuting. But then $\prod_{i \in I}^\kappa B_i$ cannot be $\text{Hom}_*\text{-}\kappa$ -commuting, for otherwise, its quotient $M_{\mathbb{N}}$ would have to be such as well, but this is impossible by (2) of this proposition.

(5) The first claim follows from statements (0-3) of this proposition; the second claim follows from the fact that (say for a countable index set) a countable direct sum of $\text{Hom}_*\text{-}\kappa$ -commuting objects is the union of an ascending chain of its proper subobjects (and by Proposition 6, not $\text{Hom}_*\text{-}\kappa$ -commuting),

(6) This statement is fully categorical and is straightforward.

Our present effort concentrates on

Task 3. Give fairly detailed account of how the classes \mathcal{H}_*^κ and $\mathcal{H}_*^{\kappa'}$ relate, for different cardinals κ and κ' .

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